

# Perturbation foundation of $q$ -deformed dynamics

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**Abstract.** In the  $q$ -deformed theory the perturbation approach can be expressed in terms of two pairs of undeformed position and momentum operators. There are two configuration spaces. Correspondingly there are two  $q$ -perturbation Hamiltonians; one originates from the perturbation expansion of the potential in one configuration space, the other one originates from the perturbation expansion of the kinetic energy in another configuration space. In order to establish a general foundation of the  $q$ -perturbation theory, two perturbation equivalence theorems are proved. The first is Equivalence Theorem I: Perturbation expressions of the  $q$ -deformed uncertainty relations calculated by two pairs of undeformed operators are the same, and the two  $q$ -deformed uncertainty relations undercut Heisenberg's minimal one in the same style. The general Equivalence Theorem II is: for any potential (regular or singular) the expectation values of two  $q$ -perturbation Hamiltonians in the eigenstates of the undeformed Hamiltonian are equivalent to all orders of the perturbation expansion. As an example of singular potentials the perturbation energy spectra of the  $q$ -deformed Coulomb potential are studied.

In searching for new physics at the extremely small space scale, motivated by recent interest of new field theoretical models and quantum theories of gravity, there are studies of quantum theories in non-commutative spaces. The realization of such quantum theories has different approaches. In one approach the  $q$ -deformed quantum theory, as a possible modification of the ordinary quantum theory at space scales much smaller than  $10^{-18}$  cm, has attracted attention. In the literature different frameworks of  $q$ -deformed quantum theories were established [1–21]. We work in the framework of the  $q$ -deformed Heisenberg algebra developed in [2,4], which is self-consistent and shows an interesting physical content. In this framework the characteristics of the dynamics and the uncertainty relations of  $q$ -deformed quantum mechanics are explored [1–6,14–21].

Perturbation  $q$ -deformed dynamics are involved. The reason is that there are two pairs of undeformed variables  $(\hat{x}, \hat{p})$  and  $(\tilde{x}, \tilde{p})$ , and two natural representations of the  $q$ -deformed operators in terms of their undeformed counterparts [2,4]. Correspondingly there are two  $q$ -perturbation Hamiltonians: one originates from the perturbation expansion of the potential in the  $(\hat{x}, \hat{p})$  system, the other originates from the perturbation expansion of the kinetic energy in the  $(\tilde{x}, \tilde{p})$  system [14,16,18,19]. At the level of operators these two  $q$ -perturbation Hamiltonians are different. In the examples of the harmonic-oscillator potential and the Morse potential, calculations showed that the expectation values of the two  $q$ -perturbation Hamiltoni-

ans in the eigenstates of the undeformed Hamiltonian are equivalent [18]. In [19] an equivalence theorem for regular potentials is demonstrated.

The two pairs of undeformed variables  $(\hat{x}, \hat{p})$  and  $(\tilde{x}, \tilde{p})$  are related by a non-trivial transformation [2,4]. It should be emphasized that this transformation is not a unitary transformation in a Hilbert space. Though it maintains the commutation relations  $[\hat{x}, \hat{p}]$ , it is not clear whether it leads to the same physical consequences in the general case.

In order to establish the foundation of the  $q$ -perturbation theory, in this paper we demonstrate two equivalence theorems for general cases. The Equivalence Theorem I states that the perturbation expressions of  $q$ -deformed uncertainty relations calculated in the  $(\hat{x}, \hat{p})$  system and the  $(\tilde{x}, \tilde{p})$  system are the same, and the two  $q$ -deformed uncertainty relations undercut Heisenberg's minimal one in the same style.

The Equivalence Theorem II states that for *any* potential (regular or singular) the expectation values of the two  $q$ -perturbation Hamiltonians in the eigenstates of the undeformed Hamiltonian are equal to all orders of the perturbation expressions. Besides regular potentials demonstrated before [18,19], as an example of singular potentials the  $q$ -deformed Coulomb potential is studied in detail.

In the following we first review the background. In terms of the  $q$ -deformed phase space variables, the position operator  $X$  and the momentum operator  $P$ , the following  $q$ -deformed Heisenberg algebra has been developed [2,4]:

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$$\begin{aligned} q^{1/2}XP - q^{-1/2}PX &= iU, \\ UX &= q^{-1}XU, \\ UP &= qPU, \end{aligned} \tag{1}$$

where  $X$  and  $P$  are hermitian and  $U$  is unitary:  $X^\dagger = X$ ,  $P^\dagger = P$ ,  $U^\dagger = U^{-1}$ . Compared to the Heisenberg algebra the operator  $U$  is a new member, called a scaling operator. The necessity of introducing the operator  $U$  is explained as follows.

$X$  and  $P$  being simultaneously hermitian is a delicate point in the  $q$ -deformed dynamics. The definition of the algebra (1) is based on the definition of the hermitian momentum operator  $P$ . However, if  $X$  is assumed to be a hermitian operator in a Hilbert space, the  $q$ -deformed derivative [22]

$$\partial_X X = 1 + qX\partial_X,$$

which codes the non-commutativity of space, shows that the usual quantization rule  $P \rightarrow -i\partial_X$  does not yield a hermitian momentum operator. A hermitian momentum operator  $P$  is related to  $\partial_X$  and  $X$  in a non-linear way by introducing a scaling operator  $U$  [4]:

$$\begin{aligned} U^{-1} &\equiv q^{1/2}[1 + (q - 1)X\partial_X], \\ \bar{\partial}_X &\equiv -q^{-1/2}U\partial_X, \\ P &\equiv -\frac{i}{2}(\partial_X - \bar{\partial}_X), \end{aligned}$$

where  $\bar{\partial}_X$  is the conjugation of  $\partial_X$ . The operator  $U$  is introduced in the definition of the hermitian momentum; thus it closely relates to properties of dynamics and plays an essential role in the  $q$ -deformed quantum mechanics. Non-trivial properties of  $U$  imply that the algebra (1) has a richer structure than Heisenberg's commutation relation. In the algebra (1) the parameter  $q$  is a fixed real number. It is important to distinguish different realizations of the  $q$ -algebra by different ranges of  $q$  values [23–25]. Following [2, 4] we only consider the case  $q > 1$  in this paper. The reason is that such a choice of the parameter  $q$  leads to a consistent dynamics. In the limit  $q \rightarrow 1^+$  the scaling operator  $U$  reduces to the unit operator; thus the algebra (1) reduces to Heisenberg's commutation relation. Thus defined, the hermitian momentum  $P$  leads to  $q$ -deformation effects, which are exhibited in the dynamical equations. The momentum  $P$  non-linearly depends on  $X$  and  $\partial_X$ . Thus the  $q$ -deformed Schrödinger equation is difficult to treat.

The  $q$ -deformed phase space variables  $X$ ,  $P$  and the scaling operator  $U$  can be realized in terms of two pairs of undeformed variables [4].

(I) The variables  $\hat{x}$ ,  $\hat{p}$  of the ordinary quantum mechanics, where  $\hat{x}$ ,  $\hat{p}$  satisfy  $[\hat{x}, \hat{p}] = i$ ,  $\hat{x} = \hat{x}^\dagger$ ,  $\hat{p} = \hat{p}^\dagger$ . The  $q$ -deformed operators  $X$ ,  $P$  and  $U$  are related to  $\hat{x}$ ,  $\hat{p}$  as follows:

$$X = \frac{\left[ \hat{z} + \frac{1}{2} \right]}{\hat{z} + \frac{1}{2}} \hat{x}, \quad P = \hat{p}, \quad U = q^{\hat{z}}, \quad \hat{z} = -\frac{i}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}), \tag{2}$$

where  $[A]$  is the  $q$ -deformation of  $A$ , defined by  $[A] \equiv (q^A - q^{-A})/(q - q^{-1})$ . It is easy to check that  $X$ ,  $P$  and  $U$  satisfy the algebra (1).

(II) The variables  $\tilde{x}$  and  $\tilde{p}$  of an undeformed algebra, which are obtained by a transformation of  $\hat{x}$  and  $\hat{p}$ :

$$\tilde{x} = \hat{x}F^{-1}(\hat{z}), \quad \tilde{p} = F(\hat{z})\hat{p}, \quad F^{-1}(\hat{z}) = \frac{\left[ \hat{z} - \frac{1}{2} \right]}{\hat{z} - \frac{1}{2}}. \tag{3}$$

Thus defined the variables  $\tilde{x}$  and  $\tilde{p}$  also satisfy the undeformed algebra:  $[\tilde{x}, \tilde{p}] = i$ , and  $\tilde{x} = \tilde{x}^\dagger$ ,  $\tilde{p} = \tilde{p}^\dagger$ . Thus  $\tilde{p} = -i\partial_{\tilde{x}}$ , where  $\partial_{\tilde{x}}\tilde{x} \equiv 1$ . The  $q$ -deformed operators  $X$ ,  $P$  and  $U$  are related to  $\tilde{x}$  and  $\tilde{p}$  as follows:

$$X = \tilde{x}, \quad P = F^{-1}(\tilde{z})\tilde{p}, \quad U = q^{\tilde{z}}, \quad \tilde{z} = -\frac{i}{2}(\tilde{x}\tilde{p} + \tilde{p}\tilde{x}), \tag{4}$$

where  $F^{-1}(\tilde{z})$  is defined by (3) for the variables  $(\tilde{x}, \tilde{p})$ . From (3) and (4) it follows that thus defined  $X$ ,  $P$  and  $U$  also satisfy the algebra (1), and (4) is equivalent to (2).

The  $q$ -deformed phase space  $(X, P)$  governed by the  $q$ -algebra (1) is a  $q$ -deformation of the phase space  $(\hat{x}, \hat{p})$  of ordinary quantum mechanics; thus, all machinery of ordinary quantum mechanics can be applied to  $q$ -deformed quantum mechanics. This means that the dynamical equations of a quantum system are the same for the undeformed phase space variables  $(\hat{x}, \hat{p})$ ,  $(\tilde{x}, \tilde{p})$  and for the  $q$ -deformed phase space variables  $(X, P)$ ; that is, the  $q$ -deformed Hamiltonian with the potential  $V(X)$  is  $H(X, P) = P^2/(2\mu) + V(X)$ .

Now we consider the perturbation treatment of this  $q$ -deformed theory. In view of the success of ordinary quantum mechanics the effects of the  $q$ -deformation must be extremely small; the perturbation investigation of the  $q$ -deformed dynamics is meaningful, and the parameter  $q$  must be extremely close to one. So we can let  $q = e^f = 1 + f + f^2$ , with  $0 < f \ll 1$ . It is accurate enough to the order  $f^2$  in the perturbation treatment.

In the  $(\hat{x}, \hat{p})$  system and the  $(\tilde{x}, \tilde{p})$  system from (2) and (4), to the order  $f^2$ , it follows that the perturbation expansions of  $X$  and  $P$  are

$$X = \hat{x} + f^2g(\hat{x}, \hat{p}), \quad g(\hat{x}, \hat{p}) = -\frac{1}{6}(1 + \hat{x}\hat{p}\hat{x}\hat{p})\hat{x}; \tag{5}$$

$$P = \tilde{p} + f^2h(\tilde{x}, \tilde{p}), \quad h(\tilde{x}, \tilde{p}) = -\frac{1}{6}(1 + \tilde{p}\tilde{x}\tilde{p}\tilde{x})\tilde{p}. \tag{6}$$

The operator  $F^{-1}(\hat{z})$  defined by (3) is not unitary,  $F^{-1}(\hat{z}) \neq F^\dagger(\hat{z})$ , which is a variable transformation between the two configuration spaces; it should be distinguished from a unitary transformation in a Hilbert space. It is not clear whether two perturbation formulations in the  $(\hat{x}, \hat{p})$  system and the  $(\tilde{x}, \tilde{p})$  system are equivalent. The situation is clarified by the following two equivalence theorems.

First we consider the perturbation treatment of the  $q$ -deformed uncertainty relation.

*Perturbation Equivalence Theorem I:* The perturbation expressions of the  $q$ -deformed uncertainty relation calcu-

lated in the  $(\hat{x}, \hat{p})$  system and the  $(\tilde{x}, \tilde{p})$  system are the same.

From the algebra (1) we obtain

$$XP - PX = iG, \quad G = (U + U^\dagger)/(q^{1/2} + q^{-1/2}).$$

To the order  $f^2$  of the perturbation expansions in the  $(\hat{x}, \hat{p})$  system and the  $(\tilde{x}, \tilde{p})$  system the operator  $G$  has the same representation:  $G = 1 - (1/2)f^2\xi\rho\xi\rho$ ; here and in what follows  $(\xi, \rho)$  represents  $(\hat{x}, \hat{p})$  or  $(\tilde{x}, \tilde{p})$ . The corresponding  $q$ -deformed uncertainty relation reads

$$\Delta X \cdot \Delta P \geq \frac{1}{2}|\langle G \rangle| \geq \frac{1}{2} - \frac{1}{4}f^2|\langle \xi\rho\xi\rho \rangle|. \quad (7)$$

Let us now consider the *undercutting phenomenon*. The Equivalence Theorem I shows that the  $q$ -deformed uncertainty relation essentially deviates from the Heisenberg one: for the case  $\Delta X \cdot \Delta P = (1/2) - (1/4)f^2|\langle \xi\rho\xi\rho \rangle|$  the Heisenberg minimal uncertainty relation  $\Delta X \cdot \Delta P = 1/2$  is undercut in the same style in the two perturbation formulations.

Now we consider the perturbation treatment of singular potentials. As an example, we study the Coulomb potential in detail.

In the  $(\hat{x}, \hat{p})$  system the definition of the  $q$ -deformed Coulomb potential is involved. Here we give its perturbation definition. Because of  $f \ll 1$  we have  $f^2\|g(\hat{x}, \hat{p})\| < \|\hat{x}\|$  where  $\|A\|$  is the norm of the operator  $A$ . In the perturbation expansion, to the order  $f^2$ , the  $q$ -deformed Coulomb potential is defined by

$$V(X) = \begin{cases} -\kappa/[\hat{x} + f^2g(\hat{x}, \hat{p})] & \text{if } \hat{x} > 0, \\ -\kappa/[-\hat{x} + f^2g(-\hat{x}, -\hat{p})] & \text{if } \hat{x} < 0, \end{cases} \quad (8)$$

where  $\kappa > 0$ . In the limit  $q \rightarrow 1^+$  the above  $q$ -deformed Coulomb potential reduces to the undeformed one  $V(\hat{x}) = -\kappa|\hat{x}|^{-1}$ . For singular potentials we use the following operator equation to treat the perturbation expansion:

$$\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A}B\frac{1}{A} + \frac{1}{A}B\frac{1}{A}B\frac{1}{A} - \frac{1}{A}B\frac{1}{A}B\frac{1}{A}B\frac{1}{A} + \dots,$$

where the norms of the operators  $A$  and  $B$  satisfy  $\|B\| < \|A\|$ . Using (2) and carefully considering the ordering between the non-commutative quantities  $\hat{x}$  and  $g(\hat{x}, \hat{p})$  in the perturbation expansion, to the order  $f^2$ , we express the  $q$ -deformed Hamiltonian of the Coulomb system by the undeformed variables  $(\hat{x}, \hat{p})$  as  $H(X, P) = H_{\text{un}}(\hat{x}, \hat{p}) + \hat{H}_{\text{I,C}}^{(q)}(\hat{x}, \hat{p})$ , where the perturbation Hamiltonian

$$\hat{H}_{\text{I,C}}^{(q)}(\hat{x}, \hat{p}) = \begin{cases} \hat{H}_{\text{I}^+}^{(q)}(\hat{x}, \hat{p}) & \text{if } \hat{x} > 0, \\ \hat{H}_{\text{I}^-}^{(q)}(\hat{x}, \hat{p}) & \text{if } \hat{x} < 0, \end{cases} \quad (9)$$

and

$$\hat{H}_{\text{I}^+}^{(q)}(\hat{x}, \hat{p}) = -\frac{1}{6}\kappa f^2 \left( \frac{1}{\hat{x}} - i\hat{p} + \hat{x}\hat{p}^2 \right) \quad (\hat{x} > 0);$$

$$\hat{H}_{\text{I}^-}^{(q)}(\hat{x}, \hat{p}) = \hat{H}_{\text{I}^+}^{(q)}(-\hat{x}, -\hat{p}) \quad (\hat{x} < 0). \quad (10)$$

In the  $(\tilde{x}, \tilde{p})$  system the  $q$ -deformed potentials have the same representations as the undeformed ones,  $V(X) = V(\tilde{x}) = -\kappa/|\tilde{x}|$ . But the momentum operator  $P$  is a non-linear function of  $(\tilde{x}, \tilde{p})$ . Using (4) and carefully considering the ordering between the non-commutative quantities  $\tilde{p}$  and  $h(\tilde{x}, \tilde{p})$  in the perturbation expansion, to the order  $f^2$ , it follows that the  $q$ -deformed Hamiltonian  $H(X, P) = H_{\text{un}}(\tilde{x}, \tilde{p}) + \tilde{H}_{\text{I,C}}^{(q)}(\tilde{x}, \tilde{p})$ , where the perturbation Hamiltonian is

$$\tilde{H}_{\text{I,C}}^{(q)}(\tilde{x}, \tilde{p}) = -\frac{f^2}{12\mu} [2\tilde{x}^2\tilde{p}^4 - 8i\tilde{x}\tilde{p}^3 - 3\tilde{p}^2]. \quad (11)$$

In the above the undeformed Hamiltonian is  $H_{\text{un}}(\xi, \rho) = \rho^2/(2\mu) - \kappa/|\xi|$ .

The two perturbation Hamiltonians  $\hat{H}_{\text{I,C}}^{(q)}(\hat{x}, \hat{p})$  and  $\tilde{H}_{\text{I,C}}^{(q)}(\tilde{x}, \tilde{p})$  originate, separately, from the perturbation expansions of the potential and the kinetic energy. At the operator level they are different. Now we show that their contributions to the perturbation shifts of the energy spectrum of the undeformed Hamiltonian in the  $(\hat{x}, \hat{p})$  system and the  $(\tilde{x}, \tilde{p})$  system are the same.

It is well known that for the undeformed one-dimensional Coulomb system [26] all the excited bound states are twofold degenerate, having an even and an odd wave function for each eigenvalue, except for the ground state which is an even state localized at the point  $\hat{x} = 0$  and having infinite binding energy. The even state  $\psi_{n+}$  and the odd state  $\psi_{n-}$  are

$$\psi_{n\pm}(\hat{x}) = \begin{cases} \psi_n(\hat{x}) & \text{if } \hat{x} > 0, \\ \pm\psi_n(-\hat{x}) & \text{if } \hat{x} < 0, \end{cases} \quad (12)$$

where

$$\psi_n(\hat{x}) = \hat{x}e^{-\hat{x}/n}F(1-n, 2, 2\hat{x}/n),$$

and  $F(1-n, 2, x)$  is the usual confluent hypergeometric function.

Now we calculate the energy shifts in the  $(\hat{x}, \hat{p})$  system contributed by the Hamiltonian  $\hat{H}_{\text{I,C}}^{(q)}(\hat{x}, \hat{p})$ . From (9), (10) and (12) it follows that for the even and the odd state the perturbation shifts of the undeformed spectrum are

$$\begin{aligned} \Delta\hat{E}_n^{(q)} &= \int_{-\infty}^{\infty} d\hat{x}\psi_{n\pm}^{(0)*}(\hat{x})\hat{H}_{\text{I,C}}^{(q)}(\hat{x}, \hat{p})\psi_{n\pm}^{(0)}(\hat{x}) \\ &= \int_{-\infty}^0 d\hat{x}(\pm\psi_n^{(0)*}(-\hat{x}))\hat{H}_{\text{I}^-}^{(q)}(\hat{x}, \hat{p})(\pm\psi_n^{(0)}(-\hat{x})) \\ &\quad + \int_0^{\infty} d\hat{x}\psi_n^{(0)*}(\hat{x})\hat{H}_{\text{I}^+}^{(q)}(\hat{x}, \hat{p})\psi_n^{(0)}(\hat{x}) \\ &= 2 \int_0^{\infty} d\hat{x}\psi_n^{(0)*}(\hat{x})\hat{H}_{\text{I}^+}^{(q)}(\hat{x}, \hat{p})\psi_n^{(0)}(\hat{x}) \\ &= -\frac{\kappa f^2}{3} \int_0^{\infty} d\hat{x}\psi_n^{(0)*}(\hat{x}) \left\{ \frac{1}{\hat{x}} - i\hat{p} + \hat{x}\hat{p}^2 \right\} \psi_n^{(0)}(\hat{x}). \end{aligned} \quad (13)$$

Similarly, in the  $(\tilde{x}, \tilde{p})$  system the energy shifts contributed by the Hamiltonian  $\tilde{H}_{I,C}^{(q)}(\tilde{x}, \tilde{p})$  in (11) are

$$\begin{aligned}\Delta\tilde{E}_n^{(q)} &= \int_{-\infty}^{\infty} d\tilde{x}\psi_{n\pm}^{(0)*}(\tilde{x})\tilde{H}_{I,C}^{(q)}(\tilde{x}, \tilde{p})\psi_{n\pm}^{(0)}(\tilde{x}) \\ &= 2 \int_0^{\infty} d\tilde{x}\psi_n^{(0)*}(\tilde{x})\tilde{H}_{I,C}^{(q)}(\tilde{x}, \tilde{p})\psi_n^{(0)}(\tilde{x}) \\ &= -\frac{f^2}{6\mu} \int_0^{\infty} d\tilde{x}\psi_n^{(0)*}(\tilde{x}) \\ &\quad \times \{2\tilde{x}^2\tilde{p}^4 - 8i\tilde{x}\tilde{p}^3 - 3\tilde{p}^2\}\psi_n^{(0)}(\tilde{x}).\end{aligned}\quad (14)$$

In the undeformed stationary states  $|\psi^{(0)}\rangle$  the time derivative of the expectation value of the operator  $\xi^m\rho^n$  is

$$\begin{aligned}i\frac{d}{dt}\langle\psi^{(0)}|\xi^m\rho^n|\psi^{(0)}\rangle \\ = \langle\psi^{(0)}|\left[\xi^m\rho^n, \frac{1}{2\mu}\rho^2 + V(\xi)\right]|\psi^{(0)}\rangle = 0.\end{aligned}$$

For the case  $m+n = \text{even}$  the above equation reduces to

$$\int_0^{\infty} d\xi\psi_n^{(0)*}(\xi)\left[\xi^m\rho^n, \frac{1}{2\mu}\rho^2 + V(\xi)\right]\psi_n^{(0)}(\xi) = 0. \quad (15)$$

From (15) for the cases of  $m=n=3$  and  $m=n=2$  it follows that for the Coulomb potential we have

$$\begin{aligned}\int_0^{\infty} d\xi\psi_n^{(0)*}(\xi)\xi^2\rho^4\psi_n^{(0)}(\xi) \\ = \int_0^{\infty} d\xi\psi_n^{(0)*}(\xi)\left[i\xi\rho^3 + \kappa\mu\left(\xi\rho^2 + 2i\rho - \frac{2}{\xi}\right)\right]\psi_n^{(0)}(\xi); \\ \int_0^{\infty} d\xi\psi_n^{(0)*}(\xi)\xi\rho^3\psi_n^{(0)}(\xi) \\ = \int_0^{\infty} d\xi\psi_n^{(0)*}(\xi)\left[\frac{i}{2}\rho + \kappa\mu\left(\rho + \frac{i}{\xi}\right)\right]\psi_n^{(0)}(\xi).\end{aligned}\quad (16)$$

Using the above two equations we prove that (13) and (14) are equivalent.

In the general cases such an equivalence can be summarized as *Perturbation Equivalence Theorem II*: For any potential (regular or singular) the expectation value  $\Delta\hat{E}_n^{(q)}$  of the Hamiltonian  $\hat{H}_I^{(q)}(\hat{x}, \hat{p})$  and the expectation value  $\Delta\tilde{E}_n^{(q)}$  of the Hamiltonian  $\tilde{H}_I^{(q)}(\tilde{x}, \tilde{p})$  in the same eigenstate of the undeformed Hamiltonian are equal to all orders of the perturbation expansions. Whereas  $\hat{H}_I^{(q)}(\hat{x}, \hat{p})$  originates from the perturbation expansion of the potential in the  $(\hat{x}, \hat{p})$  system,  $\tilde{H}_I^{(q)}(\tilde{x}, \tilde{p})$  originates from the perturbation expansion of the kinetic energy in the  $(\tilde{x}, \tilde{p})$  system.

Suppose that the Schrödinger equation for the undeformed system  $H_{\text{un}}$  is solved,  $H_{\text{un}}|\psi_n^{(0)}\rangle = E_n^{(\text{un})}|\psi_n^{(0)}\rangle$ . It is obvious that the structure of the undeformed wave function  $\psi_n^{(0)}(\hat{x}_0) = \langle\hat{x}_0|\psi_n^{(0)}\rangle$  in the configuration space  $\hat{x}_0$  and the structure of the undeformed wave function  $\psi_n^{(0)}(\tilde{x}_0) = \langle\tilde{x}_0|\psi_n^{(0)}\rangle$  in the configuration space  $\tilde{x}_0$  are the same. Because of the hermitian character of  $H_{\text{un}}(\xi, \rho)$  it

is natural to assume that its eigen wave functions satisfy the completeness relations  $\int|\xi\rangle d\xi\langle\xi| = I$  in either configuration space  $\xi = \hat{x}_0$  or  $\xi = \tilde{x}_0$ .

Now the demonstration of the Equivalence Theorem II is simple. In the  $(\hat{x}, \hat{p})$  system  $H(X, P) = H_{\text{un}}(\hat{x}, \hat{p}) + \hat{H}_I^{(q)}(\hat{x}, \hat{p})$  where the  $q$ -perturbation Hamiltonian  $\hat{H}_I^{(q)}(\hat{x}, \hat{p}) \equiv V(X(\hat{x}, \hat{p})) - V(\hat{x})$  for any potential (regular or singular). Taking the expectation value of  $H(X, P)$  in the undeformed state  $|\psi_n^{(0)}\rangle$ , we have

$$\langle\psi_n^{(0)}|H(X, P)|\psi_n^{(0)}\rangle = E_n^{(\text{un})} + \langle\psi_n^{(0)}|\hat{H}_I^{(q)}(\hat{x}, \hat{p})|\psi_n^{(0)}\rangle.$$

For the second term in the right hand side of this equation projecting  $|\psi_n^{(0)}\rangle$  to the base  $|\hat{x}_0\rangle$  and using the completeness relation  $\int|\hat{x}_0\rangle d\hat{x}_0\langle\hat{x}_0| = I$ , it leads to

$$\begin{aligned}\int d\hat{x}_0\langle\psi_n^{(0)}|\hat{x}_0\rangle\langle\hat{x}_0|\hat{H}_I^{(q)}(\hat{x}, \hat{p})|\psi_n^{(0)}\rangle \\ = \int d\hat{x}_0\psi_n^{(0)*}(\hat{x}_0)\hat{H}_I^{(q)}(\hat{x}_0, -i\partial_{\hat{x}_0})\psi_n^{(0)}(\hat{x}_0).\end{aligned}$$

Thus we obtain

$$E_n = \langle\psi_n^{(0)}|H(X, P)|\psi_n^{(0)}\rangle = E_n^{(\text{un})} + \Delta\hat{E}_n^{(q)}, \quad (17)$$

$$\Delta\hat{E}_n^{(q)} = \int d\hat{x}_0\psi_n^{(0)*}(\hat{x}_0)\hat{H}_I^{(q)}(\hat{x}_0, -i\partial_{\hat{x}_0})\psi_n^{(0)}(\hat{x}_0). \quad (18)$$

In the  $(\tilde{x}, \tilde{p})$  system

$$H(X, P) = H_{\text{un}}(\tilde{x}, \tilde{p}) + \tilde{H}_I^{(q)}(\tilde{x}, \tilde{p})$$

where the  $q$ -perturbation Hamiltonian

$$\tilde{H}_I^{(q)}(\tilde{x}, \tilde{p}) \equiv (1/2\mu)P^2(\tilde{x}, \tilde{p}) - (1/2\mu)\tilde{p}^2.$$

By a similar procedure we obtain

$$E_n = \langle\psi_n^{(0)}|H(X, P)|\psi_n^{(0)}\rangle = E_n^{(\text{un})} + \Delta\tilde{E}_n^{(q)}, \quad (19)$$

$$\Delta\tilde{E}_n^{(q)} = \int d\tilde{x}_0\psi_n^{(0)*}(\tilde{x}_0)\tilde{H}_I^{(q)}(\tilde{x}_0, -i\partial_{\tilde{x}_0})\psi_n^{(0)}(\tilde{x}_0). \quad (20)$$

From (17) to (20) we conclude that to all orders of the perturbation expansions

$$\Delta\hat{E}_n^{(q)} = \Delta\tilde{E}_n^{(q)}. \quad (21)$$

In the above the perturbation Hamiltonian  $\tilde{H}_I^{(q)}(\tilde{x}, \tilde{p})$  itself is potential independent; as for any potential it keeps the same representation, but the undeformed wave functions  $\psi_n^{(0)}(\tilde{x}_0)$  are potential dependent, thus the  $q$ -perturbation shifts  $\Delta\tilde{E}_n^{(q)}$  of the undeformed energy spectrum in the  $(\tilde{x}, \tilde{p})$  system are potential dependent.

In the  $q$ -deformed quantum theory, unlike ordinary quantum theory, there is a non-trivial transformation among two pairs of the undeformed variables  $(\hat{x}, \hat{p})$  and  $(\tilde{x}, \tilde{p})$ . It is not a unitary transformation in a Hilbert space. Such a variable transformation leads to two formulations in two configuration spaces. The  $q$ -perturbation quantum theory is much more complex than the ordinary one. The

Equivalence Theorems I and II clarify the foundation for perturbation calculations in the  $q$ -deformed dynamics. Based on the equivalence theorems the perturbation effects can be calculated in the  $(\hat{x}, \hat{p})$  system or the  $(\tilde{x}, \tilde{p})$  system. In the  $(\tilde{x}, \tilde{p})$  system for any potential the perturbation Hamiltonian  $\tilde{H}_I^{(a)}(\tilde{x}, \tilde{p})$  keeps the same form; thus it provides a unified formulation for calculating the  $q$ -perturbation shifts of the energy spectrum.

If the  $q$ -deformed quantum theory is a relevant theory for the extremely short space scale, its corrections to the ordinary quantum theory must be extremely small in the energy range of present experiments. Perturbation studies of the  $q$ -deformed dynamics show a clear indication of  $q$ -deformed modifications to the ordinary quantum theory. The investigation in the  $q$ -squeezed state [17] may provide some evidence about such  $q$ -deformed effects to present experiments. Further exploration of the effects of the  $q$ -deformation based on the  $q$ -deformed equivalence theorems is in progress.

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## References

1. J. Schwenk, J. Wess, Phys. Lett. B **291**, 273 (1992)
2. A. Hebecker, S. Schreckenberger, J. Schwenk, W. Weich, J. Wess, Z. Phys. C **64**, 355 (1994)
3. A. Lorek, J. Wess, Z. Phys. C **67**, 671 (1995) [q-alg/9502007]
4. M. Fichtmüller, A. Lorek, J. Wess, Z. Phys. C **71**, 533 (1996) [hep-th/9511106]
5. A. Lorek, A. Ruffing, J. Wess, Z. Phys. C **74**, 369 (1997) [hep-th/9605161]
6. B.L. Cerchiai, R. Hinterding, J. Madore, J. Wess, Eur. Phys. J. C **8**, 547 (1999) [math-qa/9809160]
7. A.J. Macfarlane, J. Phys. A **22**, 4581 (1989)
8. L.C. Biedenharn, J. Phys. A **22**, L873 (1989)
9. C.-P. Sun, H. Fu, J. Phys. A **22**, L983 (1989); M. Chaichian, P. Kulish, Phys. Lett. B **234**, 72 (1990); P.P. Kulish, E.V. Damashinsky, J. Phys. A **23**, L415 (1990); M. Arik, D.D. Coon, J. Math. Phys. **17**, 524 (1976)
10. S. Codriansky, Phys. Lett. A **184**, 381 (1993)
11. A. Kempf, J. Math. Phys. **35**, 4483 (1994) [hep-th/9311147]
12. R.J. McDermott, A.I. Solomon, J. Phys. A **27**, L15 (1994)
13. C.A. Nelson, M.H. Fields, Phys. Rev. A **51**, 2410 (1995)
14. Jian-zu Zhang, Phys. Lett. B **440**, 66 (1998)
15. Jian-zu Zhang, Phys. Lett. A **262**, 125 (1999)
16. Jian-zu Zhang, Phys. Lett. B **477**, 361 (2000); Chin. Phys. Lett. **17**, 91 (2000)
17. P. Osland, Jian-zu Zhang, Ann. Phys. (N.Y.) **290**, 45 (2001) [hep-th/0010176]
18. Jian-zu Zhang, P. Osland, Euro. Phys. J. C **20**, 393 (2001) [hep-th/0102014]
19. Jian-zu Zhang, Phys. Lett. B **517**, 210 (2001) [hep-th/0108111]
20. Jian-zu Zhang, Phys. Lett. B **539**, 162 (2002) [hep-th/0205125]
21. Jian-zu Zhang, Phys. Lett. B **554**, 223 (2003) [hep-th/0301007]
22. J. Wess, B. Zumino, Nucl. Phys. Proc. Suppl. B **18**, 302 (1991)
23. C. Zachos, Contem. Math. **134**, 351 (1992)
24. C.A. Nelson, M.G. Gartley, J. Phys. A **27**, 3857 (1994)
25. A.I. Solomon, J. Katriel, J. Phys. A **26**, 5443 (1993)
26. R. Loudon, Amer. J. Phys. **27**, 649 (1959)